# Coherent and incoherent trapping of a diffusion-assisted system in the presence of an external periodic field

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Field induced trapping on a line segment of a diffusion-driven system is studied with an aim to gain an insight into the occurrence of coherent stochastic resonance, which is thoroughly explored as synchronized mean-free passages to the traps. Synchronization (coherence) between the external bias, the noise in the system and the temporal trapping events is found to attain an optimum value by increasing the forcing frequency towards the relevant resonant frequency, revealing a minimum in the nonmonotonic mean-free-passage time (MFPT) to trapping. The MFPT at a given forcing frequency, is also nonmonotonic when considered as a function of the diffusion coefficient of the medium, and reveals a maximum exhibiting the least synchronization effect (incoherent trapping).

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## I. INTRODUCTION

The effect of electric field on the diffusion-influenced escape probabilities of ion-pair systems has a long age-old history [1]. The observation of the phenomena that the diffusion-driven particle mobility towards a trap increases in the presence of an externally applied driving field and attains a maximum at a particular resonant frequency of the field, which in the literature is known as coherent stochastic resonance (CSR) [2–5] and is the manifestation of complex interplay between random noise and a deterministic periodic signal, is rather new. The basic impetus to study the fundamentals of CSR, apart from exploring the physical origin of the occurrence of such phenomena, stems from the fact that the idea behind it could eventually be utilized in the separation technology for better efficiency and resolution. Electrophoretic separation of proteins, DNA [6], the chromatographic [7], and the recently proposed elegant model of high performance chromatographic [8] separation, tuned by modulated external fields, of chemical species from a mixture with close properties are the few examples that can be mentioned in this regard. The scope and utility of the present work is to obtain an insight into the occurrence of the CSR phenomena for the diffusive motion of particles on a line segment (terminated by two absorbing traps at the boundaries) in the presence of an external periodic field.

The CSR can be viewed as one of the varieties of stochastic resonance (SR) that envisages the apparently paradoxical beneficiary effects of random noise. SR was originally proposed to account for the dynamical aspect of a global climate [9]. Soon thereafter, the notion behind the noise, which is normally thought of as an undesired interfering agent with the signal detection has been changed due to its counterintuitive constructive facets that might rule the periodicity of the primary cycle of recurrent earth's ice ages. This has prompted a wide range of studies [10] that have convincingly shown that certain levels of noise can lead to more order in the system dynamics rather than merely acting as a nuisance. Today, SR is understood in a rather widespread sense and is identified as a nonmonotonic response of the system under the combined influence of the periodic signal and the noise as a function of some characteristics of either of them [9,10], or else as a function of parameters characterizing the intrinsic time scale [8,11]. The extrema in the nonmonotonic system response usually correspond to synchronization between the influencing forces. And resonance is viewed as a point of maximum synchronization [12]. Such a broader definition of SR *per se* is also thought to include the related phenomena such as CSR, which is the concern of this work.

Under the rubric of SR, most theoretical attempts and their experimental realizations are primarily concerned with a particle confined in a monostable, bistable, or multistable potential undergoing classical motion, subject to periodic forcing of the potential well(s) and are studied in terms of residence-time distribution. The periodic forcing signal is such that it alternately raises and lowers the wells with respect to the barrier but the amplitude of the signal is insufficient to cause the particle to surmount the barrier. At a given frequency of the applied field, the addition of noise enables the particle to switch from one well to the other with nonzero probability. Increasing the noise strength further leads to increased synchronization between the noise induced hopping and the field induced fluctuation of the wells. Beyond the maximum synchronization point, an increment in the noise strength will eventually lead to trivial decrease of the signal-to-noise ratio. This results into a bell shaped curve for the system response with respect to noise strength. On the other hand, keeping the noise strength fixed and varying the forcing field frequency should also lead to a point where a maximum synchronization between them is expected as and when the necessary criterion for the synchronization [10] is satisfied. As a result, SR can be studied by varying either the field frequency or the noise strength.

Since the invent of SR concepts, it has been generally accepted that its occurrence involves three essential ingredi-

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ents, namely, a source of ''random noise,'' a deterministic input signal, and a third ingredient, the presence of a nonlinear potential term that couples the random and the periodic signal to produce the maximum system response (signal-to-noise ratio). In a linear potential system, on the other hand, the additive white noise would only lead to a trivial decrease of the signal-to-noise ratio. Based on this, it was a common belief that SR is essentially the phenomenon which occurs only with the nonlinear potential term. However, recently the SR phenomena have been reported in a periodically driven linear system in the presence of multiplicative colored noise rather than additive Gaussian noise [13]. There are, however, some systems [2–5], slightly different from those above, in which a particle undergoing diffusive motion in a line segment gets trapped at the two ends. The trapped and the untrapped states of the system can be thought of as representative of the two states of the double-well potential in SR. The influence of the externally applied periodic forcing on the trapping dynamics can be equated to the forcing of the well heights (with respect to the barrier) for the SR to be observed. Whereas, the thermal white noise (diffusion) that the particle experiences in the line segment is equivalent to the added noise that helps to surmount the barrier as it is in the case of SR. As a result, it is not surprising that the three ingredients—the traps, the diffusion, and the periodic forcing—can eventually give rise to some nonmonotonic response of the system, namely, the mean-free-passage time (MFPT) to trapping as a function of forcing frequency, much like the case of SR. To distinguish this phenomenon from the other form of SR, it has been termed in the literature ''coherent stochastic resonance'' and is also the subject of our interest in this work. In CSR, the noise (diffusion) tends to equilibrate any nonequilibrium particle distribution over the line segment. The external periodic bias, on the other hand, tends to generate a nonequlibium distribution at the line ends by forcing the particle towards the traps (a positive bias forces towards the right trap, a negative bias towards the left trap of the line). We will show that for certain noise strength the competitive interplay between these two leads to coherent trapping. Any further increase of the field periodicity at a given noise strength will result in the loss of coherence, whereas, when viewed as a function of the diffusion at a given periodicity of the field, the noise leads to incoherent trapping up to certain noise strength, beyond which the system becomes completely diffusion controlled.

In the realm of CSR [2–5], what appeared important for the CSR to be observed is the temporal profile of the externally applied field. Fletcher, Havlin, and Weiss [2] were the first to consider such a system in the presence of a sinusoidal field for an initial random distribution of particles within the confined line segment. Based on a discrete random walk approach, it was shown that the MFPT on the line segment passes through a minimum as a function of the frequency of the sinusoidal field. Further, in order to explore the possibility of occurrence of CSR in the case of a bias term in the form of a telegraphic signal, exact expressions were obtained for the MFPT of the same dynamical system in the presence of deterministic and periodic telegraphic bias [4] as well as for the case of random telegraphic force [3] modeled as dichotomous Markov noise. A common observation in these studies [3,4] was that the MFPT is an increasing function of the frequency of the applied field until a sufficiently high frequency is reached when the system's response becomes uninfluenced by the applied force for all practical purposes. Thus, an occurrence of resonant like behavior (when viewed as a function of the forcing frequency) is ruled out in the case of two absorbing boundaries for a telegraphic bias, be it random or deterministic. This has prompted another analytic study [5] for the same dynamical system subject to a multistep periodic forcing, resulting in the reoccurrence of a resonant behavior. These studies involving a sinusoidal bias [2], and its single-step [4] or multistep [5] telegraphic approximations are closely related in spirit to the periodicity of the forcing fields and thus the periodicity factor could not be the origin of the occurrence of CSR. Rather, what appeared important for the CSR to be observed is the temporal shape of the field, especially at the early time scale: for an increasing bias during the first half cycle period of the field, CSR is expected to be observed, whereas a constant bias of maximum magnitude during this time will prevent any CSR from being observed. Here, we will show that the occurrence of CSR as a function of the forcing frequency is indeed a bona fide resonance [12] that leads to a coherent trapping of the particle. In addition, we will present the case where the MFPT as a function of noise strength (diffusion) can also give rise to nonmonotonic variation of the MFPT albeit the trapping in such situation is only incoherent. Thus, we will show that a nonmonotonic MFPT can appear by varying either the field frequency or the noise strength, very similarly to what has been shown in the case of SR involving metastable systems [9,10]. Consequently, the nonmonotonic behavior as a function of diffusion might also be utilized for electrophoretic separation purposes in the same spirit as it is in the case of a resonant forcing frequency.

In contrast to the separate elegant analytical approaches to the same problem involving random external force [3], periodic telegraphic force [4], and a multistep periodic force [5], our approach here is numerical. This allows one to consider a variety of forcing fields by utilizing the same algorithm and also eventually has allowed us to analyze in a greater detail the spatiotemporal profiles of the distribution function within the confinement. Based on this, we will show that the occurrence of CSR is an ''input-output'' synchronization event. Utilization of the same numerical recipe has allowed us to view the nonmonotonous variations of the MFPT, which might arise not only as a result of the coherent trapping but also for the incoherent one. In Sec. II, we will consider the basic theoretical formulations and will solve the corresponding field induced diffusion equation. Illustrative calculations will be presented in Sec. III to help delineate the origin of occurrence of the phenomena. In Sec. IV, we will conclude this work.

## II. THEORETICAL FORMULATION

The simplest generic model of CSR is the one-dimensional overdamped dynamical system defined by the Langevin equation

$$\dot{X} = \xi(t) + F(\Omega t), \tag{1}$$

for the state variable $X(t)$ in which $\xi(t)$ is some form of noise and $F$ is the external periodic bias that modulates the state of the system with a frequency $\Omega$. Thus, the system evolution is governed by the interplay between the noise-assisted mobility of the system particles and their periodic modulation due to the external input signal $F(\Omega t)$. The phenomenon of CSR that we are interested in is concerned with the type of periodic modulation term (namely, sinusoidal and periodic telegraphic signal) that can significantly change the noise-alone system dynamical properties with its cooperation either coherently or incoherently. The system is confined between two traps located at the two ends of a line segment, where the system particles are absorbed but otherwise survive within the confined line segment. The present system is completely defined by these specifications although for a nonlinear system where SR has been reported [9], such as a bistable potential system a deterministic force term needs to be added on the right-hand side of Eq. (1) to account for the potential system response towards the input periodic signal. Thus, in the present linear and confined system, the particles are assumed to be unbound and be represented by Eq. (1) for the stochastic system variable $X$.

Let us assume that the noise term in Eq. (1) has a vanishing mean $\langle \xi(t) \rangle = 0$, and a certain time correlation given by

$$\langle \xi(t)\xi(t') \rangle = 2D\,\delta(t-t'), \tag{2}$$

with $D$ having the dimensions of a diffusion coefficient. In the presence of this Gaussian white noise, the properties of a random variable $x(t)$, such as the time-dependent particle position, may be summarized in terms of the probability density function $\rho(x,t)$, which satisfies the one-dimensional Fokker-Planck equation given by

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} - AF(\Omega t)\frac{\partial \rho}{\partial x}. \tag{3}$$

Here, the positive constant $A$ is assumed to be unity, which is tantamount to presenting the solution of Eq. (3) in the dimensionless units. Next we match the present stochastic system with the following reaction scheme for a bimolecular reaction in order to arrive at an appealing correlation with the trapping dynamics.

$$A + B \xrightarrow{k(t)} P, \tag{4}$$

where $A$ denotes the ''reactant'' state that undergoes diffusive motion within the confinement and $B$ denotes another ''reactant'' representing the traps at any one of the two ends of the line segment. The ''product'' state, $P$ corresponds to escape after reaching the traps. $k(t)$ is the second-order time-dependent rate constant of the bimolecular reaction scheme. The escape of the reactant $A$ at the boundaries may be thought to be satisfied by $\rho(x,t)$ as given by

$$\rho(0,t) = \rho(L,t) = 0, \tag{5}$$

which corresponds to the traps located at the two ends of the line segment of length $L$ and the field induced diffusive motion within the line segment is governed by Eq. (3) such that $x$ can take all positive values between 0 and $L$. In order to solve Eq. (3), we also need an initial probability distribution function at time $t=0$, for which we consider the situation in which the particles are assumed to be set in motion starting at $x_0(x_0 = L/2)$, i.e.,

$$\rho(x_0, t=0) = \delta(x - x_0). \tag{6}$$

With these initial and boundary conditions given by Eqs. (5) and (6), Eq. (3) is solved for the following choice of periodic field functions [although the numerical technique that we employ to solve Eq. (3) may also be applied to other types of periodic signals and not necessarily restricted to these fields]:
sinusoidal field [2]

$$F(\Omega t) = v \sin(\Omega t), \tag{7}$$

and the periodic telegraphic field [4]

$$F(\Omega t) = \begin{cases} +v & \text{for } t \in [2nT, (2n+1)T] \\ -v & \text{for } t \in [(2n+1)T, (2n+2)T], \end{cases} \tag{8}$$

where $v$ is a strength of the bias, which is taken to be equal to 1 and $n = 0, 1, 2, \ldots$ . The period $T_0$ of the telegraphic signal is $2T$ and its rate (frequency) is the same as that of the sinusoidal signal. Thus, $T = \pi/\Omega$ such that the period of both the field is $T_0 = 2\pi/\Omega$. Constructing the telegraphic field with such specification, eventually means that each half cycle of the sinusoidal field has been replaced by a constant bias of amplitude $\pm v$. Note that the particles starting at $x_0$ at $t=0$ experiences a positive bias during the first half period of both the fields, causing the particles first to drift towards the right hand trap until the bias changes when the surviving particles will have drifted towards the left-hand trap. In other words, a positive bias forces the particle towards the right trap and the negative bias towards the left trap. The major difference between the two fields is that in each half period of a sinusoidal field the particles experience a time varying bias, whereas for the case of a telegraphic signal it is the constant bias during its each half cycle that forces the particles. As we shall see below, this difference causes the coherent trapping of the particles in the case of sinusoidal field (and not in the case of telegraphic field) and thus allows the CSR to be observed.

Equation (3) with the associated initial and boundary conditions and the various forms of external fields can be solved numerically. We employ the Crank-Nicolson finite difference scheme [14] for this purpose, which eventually provides a set of tridiagonal matrix equations, the solution of which can be obtained with the efficient routines available [15]. Also this method is robust which provides stable, accurate, and converged results.

The simplest parameter that exhibits CSR is the MFPT that corresponds to the time to reach the traps. The probability that the free-passage time of the diffusing system (starting from $x_0$) to the traps at 0 and $L$ is greater than $t$ is obtained from the solution of Eq. (3) and is given by

$$S(t)=\int_0^L \rho(x,t)dx, \tag{9}$$

which by definition is also the survival probability of the particle distribution at time $t$. This finally gives rise to the formulation of the MFPT of the diffusing system to be directly given by

$$\langle\tau\rangle=\int_0^\infty S(t)dt. \tag{10}$$

With this definition of the particle survival probability, it follows from the phenomenological kinetic law for the reaction scheme (4) at a given concentration $\rho_B$ of the reactant $B$ that the reaction rate law is $[dS(t)/dt]=-k(t)S(t)\rho_B$. As a corollary to this, let us also define a quantity, the free-passage-time density function (FPTDF), $g(t)$ for the probability that the reaction occurs between $t$ and $t+dt$. Physically, $g(t)dt$ then represents the probability that the particle reaches the trap at the line boundaries over the time interval between $t$ and $t+dt$, and thus the reaction takes place. Thus, by definition $g(t)$ also represents the reaction rate and is given by

$$g(t)\equiv k(t)S(t)\rho_B=-\frac{dS(t)}{dt}. \tag{11}$$

Below we will utilize this definition of the FPTDF, while exploring the occurrence of CSR.

Analysis of the results requires an understanding of three relevant time scales. The first one is the MFPT $\langle\tau\rangle$, as introduced above and originates from the combined influence of the diffusion and applied periodic field. The second one is the time $\tau_D$ as it would be in the absence of periodic bias, given as $\tau_D=(L-x_0)x_0/2D$ representing the pure diffusion-driven particle residence time within the confinement before it gets trapped at the line boundaries. The third relevant time scale in this problem is the time $\tau_F$ required for the particle to be trapped at the boundaries driven by the external bias alone. It is important to note here that for such an event to take place for a $\delta$-function initial distribution of particles [cf. Eq. (6)] one must satisfy $\tau_F<T_0/2$. In other words, when this condition is satisfied, it is the first half period of the field within which the particle can be trapped (in the absence of diffusion) driven by only the external bias, failing which the particle will oscillate back and forth in line segment requiring an infinite time to be trapped. For a telegraphic field $\tau_F$ is simply given as $\tau_F=(L-x_0)/v$. For a sinusoidal field, on the other hand, $\tau_F$ can be obtained requiring that $(L-x_0)^{-1}\int_0^{\tau_F}\sin(\Omega t)dt=1$, from which one can obtain $\tau_F$ given as $\tau_F=\Omega^{-1}\cos^{-1}[1-(L-x_0)\Omega]$. Note that both $\tau_D$ and $\tau_F$ can be viewed as parameters characterizing the system under consideration (such as the confiner length, initial condition, noise strength, applied field periodicity, and its amplitude). Thus, they can be identified as causes of the consequent system dynamics, measured through $\langle\tau\rangle$. As is obvious depending on the causes (i.e., values of $\tau_D$ and $\tau_F$) there can be a variety of consequences. For example, noise level $D$ can be so low as to ensure that $\tau_D\gg\tau_F$ and the system dynamics is controlled entirely by the external field characteristics. With further increase of noise level to the extent that $\tau_F\ll\tau_D$ condition is still maintained, one can arrive at an incoherent trapping regime where the noise added increases MFPT. On the other hand, noise (diffusion) acts in coherence with the applied field and resonance occurs when $\tau_D\sim\tau_F$. This stochastic resonance leads to an enhancement of the effective passage rate. Next we will present these cases.

## III. ILLUSTRATIVE CALCULATION

Diffusion in a confinement tends to equilibrate the nonequilibrium particle distribution, if any, giving rise to a uniform distribution along the confining line segment. The applied bias directs the distribution to the traps (a positive bias towards the right and a negative bias towards the left), and thus tends to produce a nonequilibrium particle distribution. As a result, an ordered and directional trapping of the field driven particles might get disturbed or assisted interfering with the diffusion in the medium, which can be referred to as noise. The degree and the nature (coherent or incoherent) of interference will, of course, depend on many factors apart from the temporal profile of the field (e.g., sinusoidal, telegraphic), namely, the frequency (period) and strength of the applied field, strength of the noise (diffusion) in the system, the length of the line segment, and initial position of the particles in this line segment. The fate of the particles within the confiner is largely decided by these factors. For example, for a given confiner length and initial position of the particles within it and keeping other factors unaltered, one can intuitively conceive of several possible combinations of the field frequencies and the noise strengths. Their natural consequences on the system lifetime (MFPT) are depicted in Fig. 1 for the two types of field chosen. The results presented in this figure are representative of the combined influences of the noise, inherent in the system, and external periodic forcing on the irreversible trapping process, which can also be taken to represent a finite stochastic system that undergoes diffusive motion (starting from the middle of the confiner) characterized by a Gaussian white noise of strength $D$ under the influence of an external force.

The landscape of the MFPT values in the $D$-$\Omega$ space shown in this figure reestablishes the near-analytic solutions on the investigation of occurrence of CSR involving a usual telegraphic [4] and a multistep telegraphic [5] field (note that a multistep telegraphic field is another way to represent a sinusoidal field). Namely, a sinusoidal (multistep oscillating telegraphic) field can induce a coherent motion in the system, which eventually reduces the free-passage time to a minimum at a resonant frequency, beyond which the coherence is lost and the MFPT value increases. This we call the occurrence of CSR. For an usual telegraphic field, on the other hand, there is no such resonant frequency and the MFPT value for such a field is an ever increasing function of the field frequency until it converges to a pure diffusive

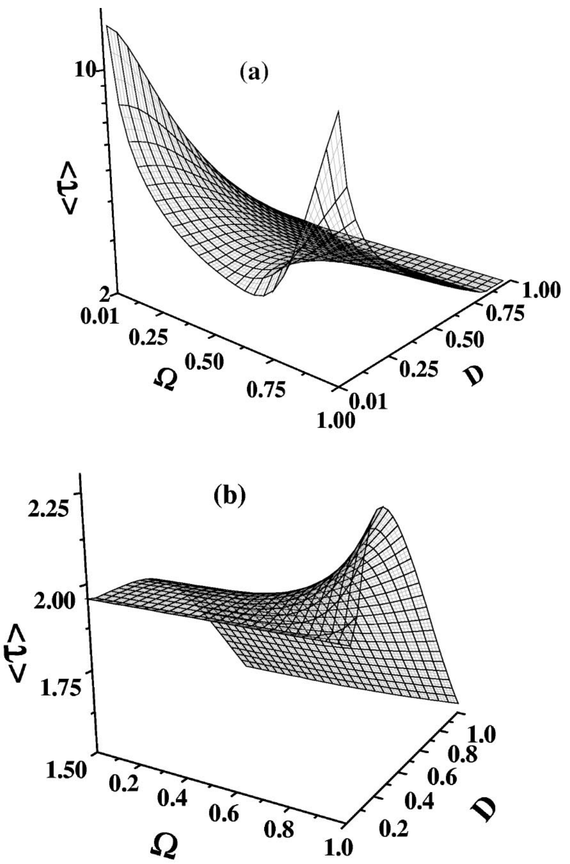


FIG. 1. The $D$-$\Omega$ landscape of mean-free-passage times $\langle\tau\rangle$ to the traps, located at the boundaries of a line segment of length 4 for the applied periodic fields; (a) sinusoidal and (b) telegraphic. All quantities are in dimensionless units.

value. In addition to these, from Fig. 1 it is also clear that the MFPT value passes through a maximum when viewed as a function of noise strength, prevailing in the medium, for certain range of values of forcing frequencies. We have also noticed that the occurrence of such a bell shaped MFPT characteristics as a function of the noise strength at certain fixed forcing frequency for other types periodic fields, such as the cosine, the sawtooth, and the ramp-wave fields (not reported here) is always inevitable. This is one of the main observations of this work, which is much reminiscent to the stochastic resonance phenomena observed in metastable systems [9], albeit occurs due to a different form of mechanism. Next we will discuss the underlying mechanism behind the occurrence of these extremes, first qualitatively and then more rigorously with the help of the calculated FPTDF values.

A systematic approach towards this goal would be to identify four extreme combinations of $D$ and $\Omega$ values in the $D$-$\Omega$ space, where these phenomena could be observed. They are, a very low and a very high value of $D$ and for each of these, a very low and a very high value of $\Omega$. For both the limits of $D$, when the forcing frequency is very high, such that $\tau_F > T_0/2$; the effect of the external bias would be merely the rapid oscillatory motion of the particle within the confiner. As a result, the trapping dynamics would be entirely diffusion limited. When both $D$ and $\Omega$ are sufficiently low (such that $\tau_F < T_0/2$), noise will act only as a small disturbance to the almost exclusively field driven system. Thus, the trapping dynamics would be external bias controlled. Finally, for the same low value of $\Omega$, $D$ could be very high which would then control the dynamics. Put in simpler terms (for the extreme values of $D$ and $\Omega$), amongst $\tau_F$ and $\tau_D$, whichever is less will eventually control the dynamics. It is only at the intermediate values of $D$ and $\Omega$, so that $\tau_F$ and $\tau_D$ are comparable, one would expect the interference between these two mechanisms leading to coherent trapping and, consequently, the effective passage rate increases. Finally, when the noise level is too low such that the system dynamics is controlled by external bias, further addition of noise (increasing $D$) could lead to increase in particles' MFPT to the traps through incoherent interference. This we denote as incoherent trapping. The spatiotemporal profiles of the probability distribution function [cf. $\rho(x,t)$ of Eq. (3)] presented in Figs. 2 and 3, and the extremes in Fig. 1 are the manifestation of these interfering regimes.

In both the Figs. 2 and 3, the system is initially thought to be at the middle of the confiner with a $\delta$-peaked Gaussian distribution. As stated above, the system has two states; the trapped and the untrapped ones. The results presented in these figures are the probability distribution function referring to the untrapped state. Thus, as the trapping proceeds, $\rho(x,t)$ decreases as a function of time. Also an unsuccessful trapping at a given boundary will result into oscillations of this distribution back into the boundary of the confiner. In Fig. 2, we present the case for a sinusoidal bias at various frequencies and at a given noise strength $(D=5)$. The corresponding characteristic time scales for the problem are indicated. Evidently, the situation in Figs. 2(a) and 2(b) follow the criterion, $\tau_F \sim \tau_D$ [more so in the case of Fig. 2(b) than in Fig. 2(a)]. As a consequence of this, the effective passage time $\langle\tau\rangle$ decreases more the more closely the value of $\tau_F$ and $\tau_D$ is matched, and thus refers to coherent trapping. As a result, at $\Omega = 0.07$ (which we call resonant frequency $\Omega_{res}$) the MFPT is the minimum. This observation can intuitively be understood: In a field-free case, the noise alone tries to randomize the initial distribution so that the spatial position of the peak of the distribution remains unchanged, whereas the width of the distribution spreads symmetrically over time. The tail of the distribution eventually gets trapped at the two identical boundaries. The time taken $\tau_D$ for trapping under this situation may be identified as the intrinsic time of the system. Whereas the field driven trapping time $\tau_F$ during the first half cycle of the field (that drives the system towards the right trap), may be identified as characteristic external time. At $\tau_F > \tau_D$ [cf. Fig. 2(a)], diffusion is strong enough so that some of the distribution can be simultaneously trapped in the left trap (escaping the initial positive bias) with the field driven trapping on the right trap. This synchronous trapping, simultaneously at the two ends, can be thought to attain a maximum closer the values of $\tau_F$ and $\tau_D$ are matched. As a result, until then $\rho(x,t)$ is not expected to show oscillatory behavior [cf. Fig. 2(a)]. Beyond this at a slightly higher frequency, as $\tau_F < \tau_D$ condition is approached, the noise and the bias together set the distribution into an oscillatory motion with effective coherent trapping at both ends. This is evident from Fig. 2(b), where at the resonant frequency, the distribu-

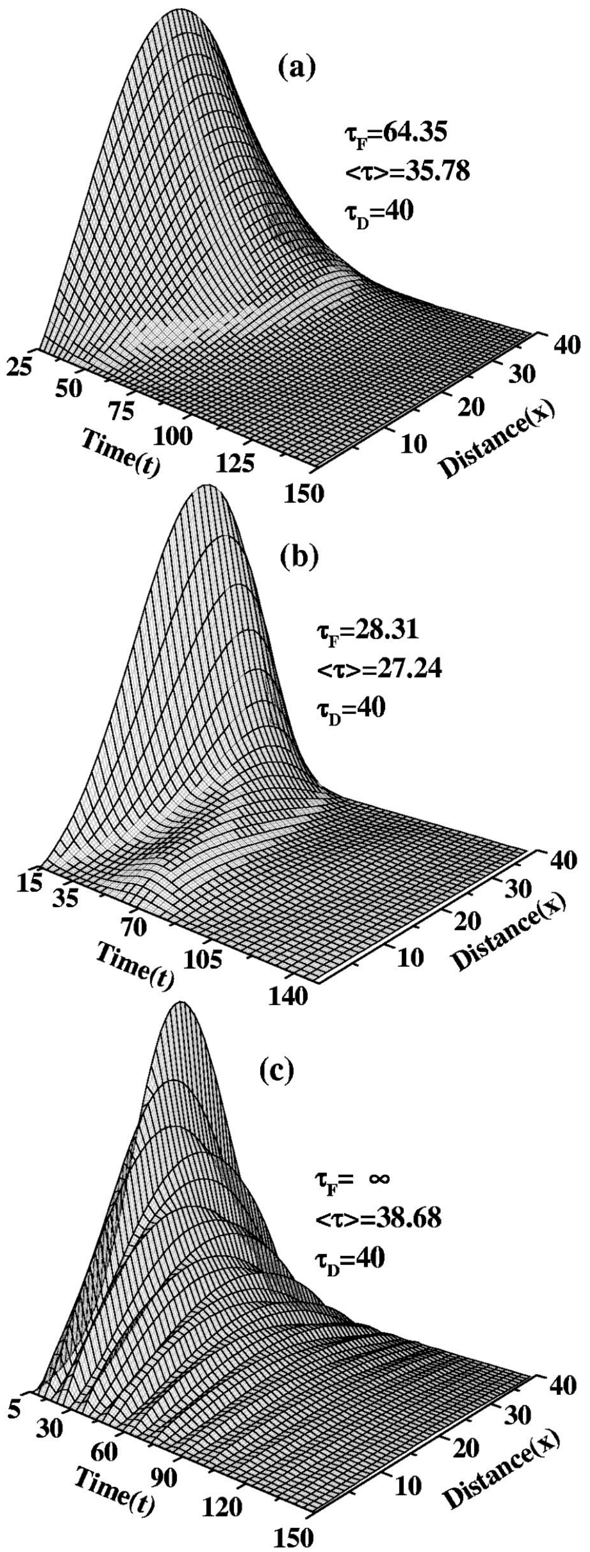


FIG. 2. Time-dependent probability density function $\rho(x,t)$ of the diffusion- ($D=5$) driven system that is initially at the middle of a linear confiner ($L=40$), terminated by two absorbing boundaries. The system is under the influence of a sinusoidal field with forcing frequencies: (a) $\Omega=0.01$, (b) $\Omega_{res}=0.07$, and (c) $\Omega=0.5$. Three characteristic time scales (dimensionless) are indicated.

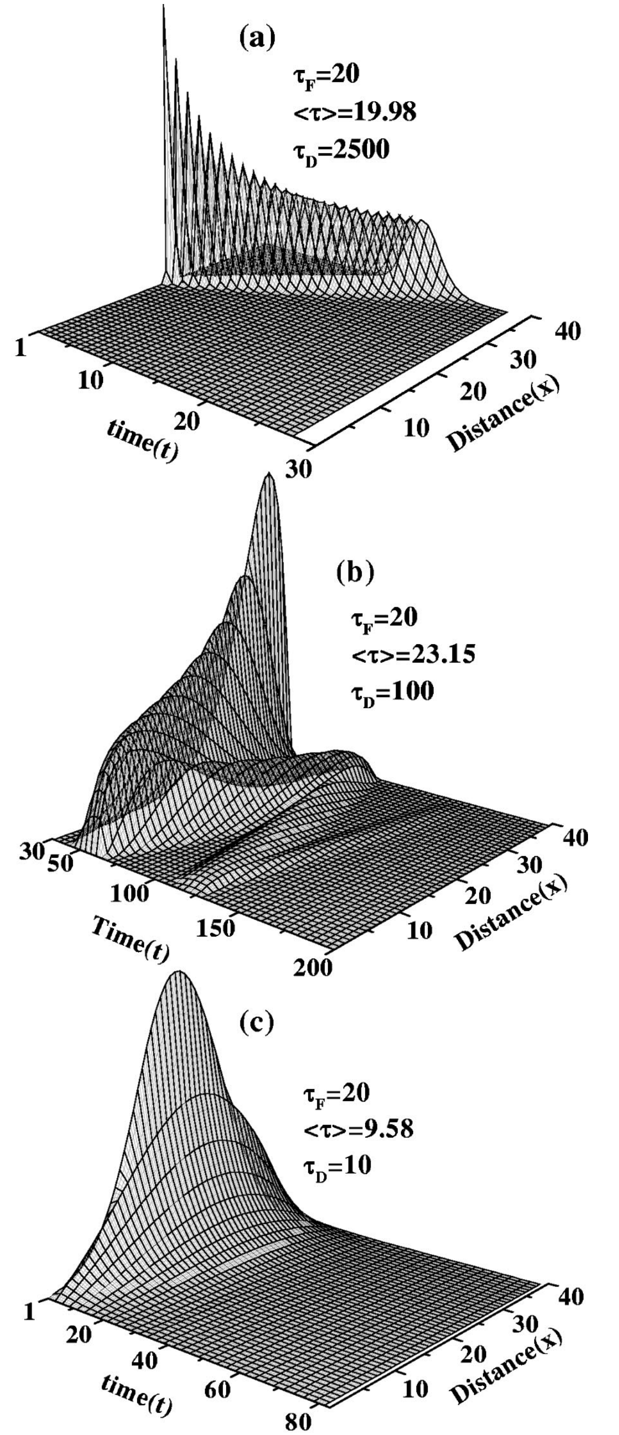


FIG. 3. Same as Fig. 4 but under the influence of a telegraphic bias ($\Omega=0.1$) and at various noise strengths (a) $D=0.08$, (b) $D=2$, and (c) $D=20$.

tion, although it could not be successfully trapped during the first half cycle of the field, experiences the successive half cycles in quick succession, leading to its effective trapping at the boundaries and thus the MFPT attains the minimum. Beyond this point, coherence worsens. Note that $\tau_F$ for a sinusoidal field is a function of $\Omega$, passes through a minimum, and has finite values within $\Omega \leqslant 2/(L-x_0)$. Beyond this, the rapidity of the bias changeover renders the system diffusion controlled such that the MFPT approaches the limits of $\tau_D$. The data presented in Fig. 2(c) is for a system close to this situation. It is important to realize here that a telegraphic field offers the maximum bias from the beginning, and thus $\tau_F$ in this case is not a function of the forcing frequency. As a result, a synchronized and coherent trapping as a function of the forcing frequency could not be observed for telegraphic field, which (as discussed above) is in contrast to a sinusoidal field.

Figure 3 depicts the $\rho(x,t)$ data for a system driven by a telegraphic field at different noise strengths. In Fig. 3(a) (when $\tau_D \gg \tau_F$), the external bias alone drives the system to the right trap such that the dynamics remains entirely field driven and thus $\langle \tau \rangle \approx \tau_F$. Further addition of noise (such that the condition, $\tau_F \ll \tau_D$ still holds) acts incoherently with the applied bias. The result presented in Fig. 3(b) is a representative case of the incoherent trapping. At this condition, the particles oscillate back and forth and the noise acts as a nuisance, resulting into a longer system lifetime. Note that in both Figs. 2(b) and 3(b) it is this oscillatory motion that is responsible for coherent and incoherent trapping, respectively. But in the former, each oscillation is a reflection of noise and bias assisted coherent motion that leads to sizable fraction of the population to be trapped, whereas in the case of incoherent trapping, although the bias drives the system to the boundaries, it is the background noise over which these oscillations build up. Even with further addition of noise to the extent that $\tau_F \gg \tau_D$, the system will again be diffusion controlled. Figure 3(c) is a testimony to this effect.

Now we will further testify to the coherent (and so the occurrence of CSR) and incoherent nature of trapping on a more rigorous ground. Before that it would be justified to summarize a bona fide resonance in the parlance of SR. In Ref. [12], with the help of an analog circuit, a continuous stochastic process exhibiting SR has been simulated into a stochastic point process and the corresponding residence times were mapped as a function of the frequency of the signal. The observations were as follows: residence times were found to form distinct peaks, which are the signature of synchronization between the two driving forces present in the system. The strength of the first peak (namely, the area under the peak) when plotted against the forcing frequency, passes through a maximum, which is identified as the resonant point; a point of maximum synchronization. Further study [16] involving a bistable potential system with one absorbing and one reflecting point, in addition to supporting these facts also showed that the FPTDF peaks are separated by the period of the field.

Following this we have presented in Fig. 4 the calculated FPTDF values as a function of a sinusoidal forcing frequency (at given $D$) and also as a function of $D$ (at a given $\Omega$) in presence of a telegraphic signal. It can be seen from Fig. 4(a) that as the frequency increases the number of FPTDF peaks increases. These peaks are separated by the half period of the field, which shows that the probability of untrapped to trapped state transition attains a maximum after each $T_0/2$ time separation. The number of peaks depends on the system survival times, MFPT. The heights of the peaks fall off exponentially with time. Each odd peak refers to the

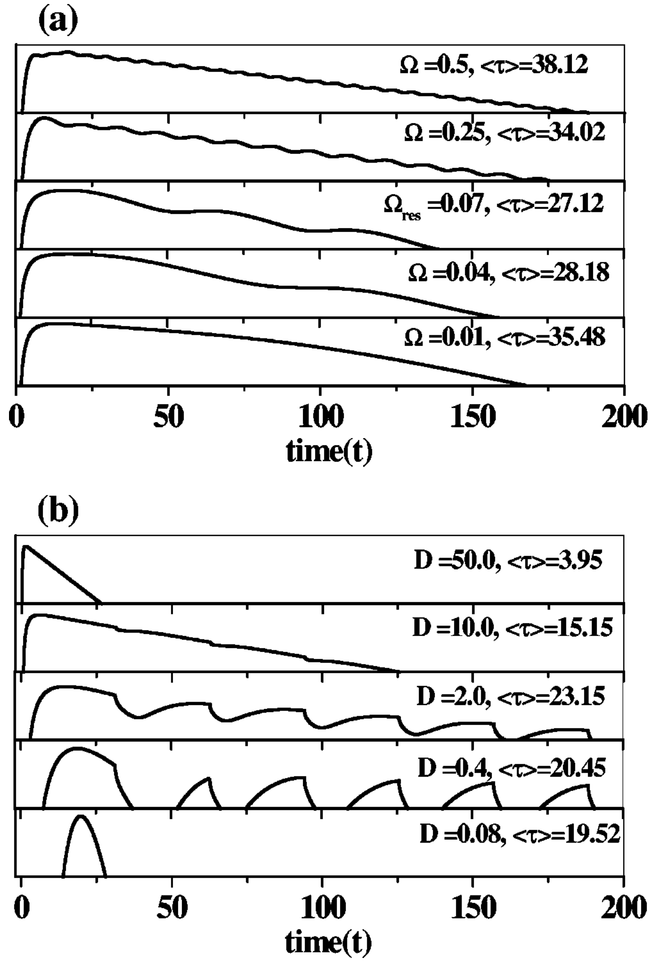


FIG. 4. The distribution of the free-passage-time density function $g(t)$ at various conditions: (a) fixed $D$ ($D=5$) at various frequencies of a sinusoidal bias and (b) at various $D$ and at a given frequency ($\Omega=0.1$) of a telegraphic bias. The length of the line segment is 40. The dimensionless values of $\Omega$ and $D$ and the corresponding MFPT values are indicated.

trapping at the right boundary and each even peak to the left boundary. All these features are the signatures of synchronized trapping events, which is similar to the case of SR [12,16]. In Fig. 4(b), the peaks are again separated by the half period of the field, but now as the noise strength increases, the background over which these peaks appear increases. This leads to longer lifetime for the system and thus more number of peaks appear within its lifetime. This is so until the point of maximum incoherence is reached (i.e., $D=2$). Beyond this, the number of peaks decreases and so the system lifetime also decreases, indicating some degree of coherent trapping, until at sufficiently high values of $D$ when diffusion alone takes control of the dynamics.

The observation that areas under each successive peak decrease can be taken to be a measure of the probability of transition; larger the area more probable is the transition. Integration of these peaks over time within the well prescribed limit give the areas $P_n = \int_{T_n - \alpha T_0}^{T_n + \alpha T_0} g(t)\,dt$, where $T_n$ is the temporal position of the $n$th peak and $\alpha$ is taken to be 0.25. The limits of the integration necessarily mean that the

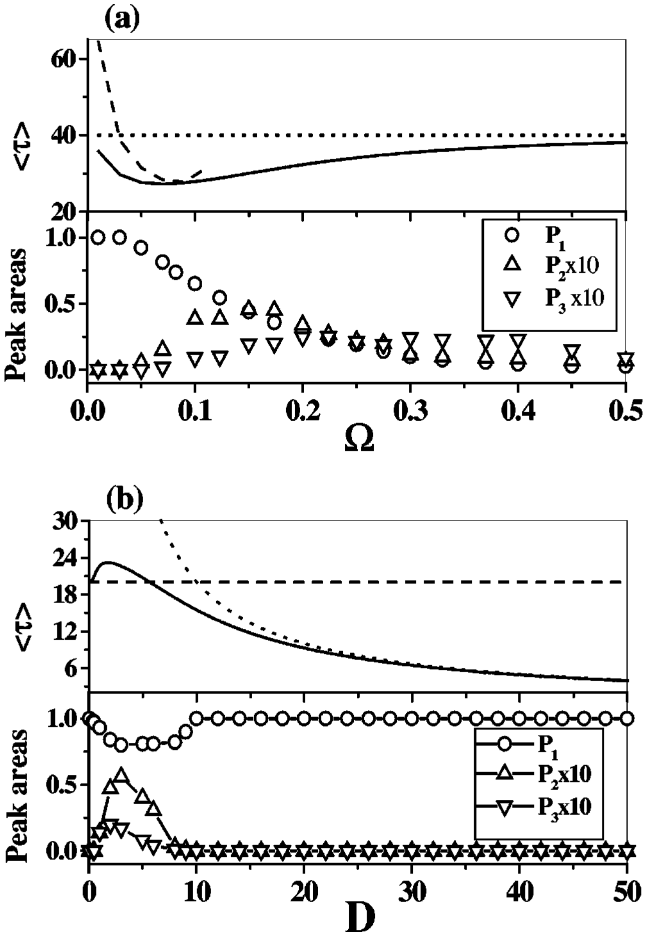


FIG. 5. The MFPT (solid line) and the peak areas of the first three FPTDF peaks plotted as a function of (a) frequency of a sinusoidal field when $D=5$ and (b) $D$ when the external bias is telegraphic with $\Omega=0.1$. $\tau_F$ (dashed line) and $\tau_D$ (dotted line) are also shown for the sake of comparison. The length of the line segment is 40. All quantities are in dimensionless units.

peak area, centered at $T_n$, is calculated for exactly half period of the field. In addition, wherever necessary (cf. Fig. 4) the exponential background of the noise is subtracted. Finding the areas in this way ensures that their values would reflect the strength of synchronization, if any. In Fig. 5, we have presented the peak areas along with the three characteristic time scales of the problem.

In Fig. 5(a), the case for a sinusoidal bias as a function of the frequency of the signal and at a given noise strength is presented. In the figure, the region where the resulting system dynamics ($\langle\tau\rangle$) is faster than either of its causes ($\tau_F$ and $\tau_D$) and, consequently, the occurrence of CSR, is clearly seen. At sufficiently low values of $\Omega$ (up to a frequency when $\tau_F=\tau_D$) just one FPTDF peak appears with its area equal to unity. This reflects the most coherent trapping during the long first half period of the field (longer than the half period of the field at resonant frequency). At slightly higher $\Omega$, the first peak area $P_1$ decreases but $P_2$ and $P_3$ start increasing. This is indicative of the fact that an unsuccessful trapping at the right boundary during the first half period will increase the probability of trapping at next successive half cycles. For example, at resonant frequency ($\Omega_{res}=0.07$) although $P_1<1$, $P_2$ and $P_3$ increase. For the chosen parameters, at resonant condition although three half periods of the sinusoidal bias are necessary to completely trap the distribution, the frequency of the bias is such that these events occur in quick succession, thus effectively reducing the system lifetime. At still higher frequencies (until the system becomes completely diffusion controlled) each unsuccessful trapping at a given boundary causes more peaks to appear but with decreasing area, rendering the system lifetime to increase. At resonant condition, the peak areas and their temporal positions become optimum such that the MFPT of the system is the minimum.

In Fig. 5(b), the peak areas and the characteristic time scales are plotted as a function of the noise strength $D$ for a telegraphic field at a given frequency. The region of incoherent trapping (when $\tau_F \ll \tau_D$), where the further addition of noise merely increases the system lifetime, can be clearly seen in the figure. At low noise the system is completely field driven ($\langle\tau\rangle=\tau_F$) so that only one FPTDF peak area appears. As the noise strength increases, $P_1$ decreases and $P_2$ and $P_3$ increase for the same given field frequency. This causes the MFPT to increase that eventually attains a maximum. After this, $P_1$ increases (and MFPT decreases), which is a reflection of the coherence. At $\tau_F=\tau_D$, the system again attains the maximum value for $P_1$ and so also is the coherence. This is also evident from the fact that the MFPT of the system is lower than either of the $\tau_F$ or $\tau_D$. At still higher diffusion coefficient, i.e., in a well-stirred system, the MFPT can be seen approaching the diffusion controlled limit. A word of caution is necessary here. Diffusion alone in the absence of a field will always drive the distribution to the symmetric boundaries, giving rise to one single FPTDF peak and the maximum area under it. Thus, going by the criterion that the FPTDF peak areas are a measure of synchronization may sometimes be proved deceptive for the type of diffusing system that we are considering. As is the case in Fig. 5(b), where at sufficiently high $D$, the system is effectively diffusion controlled but $P_1$ maintains its highest value. Nevertheless, it is an useful tool to measure the strength of synchronization, but that would be more dependable with the simultaneous knowledge of the system's characteristic time scales.

## IV. CONCLUSION

We have solved numerically the field induced diffusive dynamics of trapping of a system that undergoes one-dimensional motion in a line segment. This has allowed us to explore the occurrence of CSR in a much greater detail in terms of the mean-free-passage time to the traps. We have introduced three characteristic time scales, in the light of which the nature of interference (coherent or incoherent) between the external bias and the inherent system noise could be judged. They are, namely, the noise-free and exclusively field driven system lifetime $\tau_F$, the field-free and exclusively noise driven system lifetime $\tau_D$, and the combined noise and field driven time, the MFPT. We have shown that the nature of interference can be characterized by the characteristic time scales of its causes: $\tau_F \sim \tau_D$ (coherent trapping) and $\tau_F \ll \tau_D$ (incoherent trapping). Illustrative calculations are presented to show that the degree of interference between the two driving mechanisms depends on many factors: the temporal profile of the field (e.g., sinusoidal, telegraphic), the frequency (period), and strength of the applied field, strength of the noise (diffusion) in the system, the length of the line segment, and initial position of the particles in this line segment.

The system shows, as a function of both $\Omega$ and $D$, a nonmonotonic response, measured in terms of the MFPT to the traps. These nonmonotonic responses are further analyzed with the help of the areas under the FPTDF peaks. The extremes of the response, which are usually a measure of the synchronization between the influencing forces in the parlance of SR, have been found to have two facets; coherent and incoherent form of trapping. It has been found that the strength of synchronization is more (FPTDF peak area, $P_1$ is maximum) if the values of intrinsic $\tau_D$ and external $\tau_F$ time scales are closer. The minimum of the MFPT values as a function of $\Omega$ (in the case of a sinusoidal field) and the occurrence of CSR have been identified as the optimal of this condition and the temporal occurrence of the trapping events (cf. Figs. 4 and 5), whereas, the maximum in the system response as a function of $D$ has been identified as an outcome of the incoherent interplay between the two driving mechanisms, adjudged by the FPTDF peak areas and the condition $\tau_F \ll \tau_D$ such that further addition of noise increases the system lifetime.

The occurrence of the incoherent-trapping-related maximum as a function of the noise strength has been found to be almost universal with respect to the types of the external bias (sinusoidal, telegraphic, truncated telegraphic, cosine, sawtooth, or else the ramp-wave type). But the appearance of the minimum (as a function of $\Omega$) and its concomitant, the CSR is found to be bias specific. For CSR to appear, the external bias to start with should be an increasing function of time and $\tau_F$ a function of $\Omega$. We believe that as far as applications of the phenomena are concerned, such as the improved efficiency in the separation technology, both the extremes of the system response, which can be realized either by varying the noise intensity or else the forcing frequency, can be utilized as necessary.

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